Questions from Homework

(8) $f(3)=33$ $\boxed{f(x)=4x^2-2x+3}$

$f'(3)=22$ $f'(x)=8x-2$

$f''(3)=8$ $f''(x)=8$

(4) $f(x)=\sqrt{1+x^3} = (1+x^3)^{1/2}$

$f'(x)=\frac{1}{2}(1+x^3)^{-1/2}(3x^2) = \dfrac{3x^2}{2(1+x^3)^{1/2}}$

$$f''(x)=\frac{6x(2)(1+x^3)^{1/2}-3x^2(1+x^3)^{-1/2}(3x^2)}{\left[2(1+x^3)^{1/2}\right]^2}$$

$$f''(x)=\frac{12x(1+x^3)^{1/2}-9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$$f''(x)=\frac{3x(1+x^3)^{-1/2}\left[\overset{4+4x^3-3x^3}{4(1+x^3)-3x^3}\right]}{4(1+x^3)}$$

$$f''(x)=\frac{3x(4+x^3)}{4(1+x^3)^{3/2}} = \frac{3x(4+x^3)}{4\sqrt{(1+x^3)^3}}$$

$$f''(2)=\frac{6(\cancel{12}^{3})}{{}_{1}\cancel{4}\sqrt{729}} = \frac{\cancel{18}^{2}}{\cancel{27}_{3}} = \frac{2}{3}$$

Implicit Differentiation

So far we have described functions by expressing one variable *explicitly* in terms of another variable: for example,

$y = x^2$

$\frac{dy}{dx} = 2x$

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x .
- Examples
  - $x^2 + y^2 = 25$
  - $x^3 + y^3 = 6xy$

$x^2 + y^2 = 25$
$y^2 = 25 - x^2$
$y = \pm\sqrt{25 - x^2}$

$\cancel{(5-x)(5-x)}$
$\cancel{25 - 5x - 5x + x^2}$
$\cancel{25 - 10x + x^2}$

- The first equation could easily be rearranged for $y = \ldots$

$$y = \pm\sqrt{25 - x^2}$$ Actually gives two functions

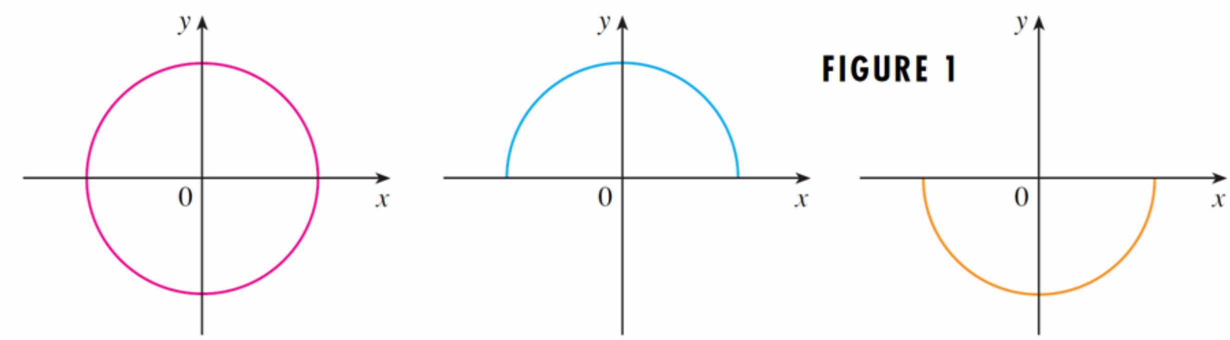

FIGURE 1

(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Implicit Differentiation

- There is a way called *implicit differentiation* to find $\boxed{dy/dx}$ without solving for y :
  1 - First differentiate both sides of the equation with respect to x ;
  2 - Then solve the resulting equation for y' . $\left(\frac{dy}{dx}\right)$
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
  a) dy/dx
  b) an equation of the tangent at the point (3, 4).

$x_1 = 3 \quad y_1 = 4$

Solution:

Start by differentiating both sides of the equation:

a) $x^2 + y^2 = 25$

$$2(x)(1) + 2(y)\left(\frac{dy}{dx}\right) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = -\frac{x}{y}$$

b) Find the equation of the tangent @ (3,4)

(i) Find m:

$$\frac{dy}{dx} = -\frac{x}{y} = \frac{-(3)}{(4)} = -\frac{3}{4} \leftarrow m$$

(ii) Find equation:

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y - 4 = -\frac{3}{4}x + \frac{9}{4}$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 4$$

$$\boxed{y = -\frac{3}{4}x + \frac{25}{4}} \text{ or } 4y = -3x + 25$$

$$\boxed{3x + 4y - 25 = 0}$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the <u>upper</u> semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate f:

(i) $y = (25 - x^2)^{1/2}$

$y' = \frac{1}{\cancel{2}}(25 - x^2)^{-1/2}(-\cancel{2}x) = \frac{-x}{\sqrt{25 - x^2}}$

(ii) $y' = \frac{-x}{\sqrt{25 - x^2}} = \frac{-(3)}{\sqrt{25 - (3)^2}} = \frac{-3}{4} \rightarrow m$

(iii) $y - y_1 = m(x - x_1)$

$y - 4 = -\frac{3}{4}(x - 3)$

$y - 4 = -\frac{3}{4}x + \frac{9}{4}$

$\boxed{y = -\frac{3}{4}x + \frac{25}{4}}$

Solution (cont'd)

- So $f'(3) = -\dfrac{3}{\sqrt{25-3^2}} = -\dfrac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.
- Note that although this problem could be done both ways, implicit differentiation was easier!

Sometimes Implicit Differentiation is not only the easiest way, it's the *only* way

Example:

Given $x^3 + y^3 = (6x)(y)$

Product

$f(x)$ $g(x)$

Find $\frac{dy}{dx}$

$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$

$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$

$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$

$\frac{dy}{dx} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)} = \boxed{\frac{2y - x^2}{y^2 - 2x}}$

Product

Given $$2x^5 + (x^4)(y) + y^5 = 36$$

Find $\frac{dy}{dx}$

$$10x^4 + 4x^3y + x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = 0$$

$$x^4\frac{dy}{dx} + 5y^4\frac{dy}{dx} = -10x^4 - 4x^3y$$

$$\frac{dy}{dx}(x^4 + 5y^4) = -10x^4 - 4x^3y$$

$$\frac{dy}{dx} = \frac{-10x^4 - 4x^3y}{x^4 + 5y^4} = \boxed{-\frac{10x^4 + 4x^3y}{x^4 + 5y^4}}$$

Homework

Exercise 2.7 Page 107
Do # 1-3, 5, 7

③ d) $(x+y)^3 = x^3+y^3$ a) $(-1,1)$

(i) $\frac{\cancel{3}(x+y)^2\left(1+\frac{dy}{dx}\right)}{\cancel{3}} = \frac{\cancel{3}x^2+\cancel{3}y^2\frac{dy}{dx}}{\cancel{3}}$

$(x+y)^2\left(1+\frac{dy}{dx}\right) = x^2+y^2\frac{dy}{dx}$

$(x^2+2xy+y^2)\left(1+\frac{dy}{dx}\right) = x^2+y^2\frac{dy}{dx}$

$x^2+2xy+y^2+x^2\frac{dy}{dx}+2xy\frac{dy}{dx}+y^2\frac{dy}{dx} = x^2+y^2\frac{dy}{dx}$

$x^2\frac{dy}{dx}+2xy\frac{dy}{dx}+\cancel{y^2\frac{dy}{dx}}-\cancel{y^2\frac{dy}{dx}} = \cancel{x^2}-\cancel{x^2}-2xy-y^2$

$x^2\frac{dy}{dx}+2xy\frac{dy}{dx} = -2xy-y^2$

$\frac{dy}{dx}(x^2+2xy) = -2xy-y^2$

$\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$

(ii) $\frac{dy}{dx} = \frac{-2(-1)(1)-(1)^2}{(-1)^2+2(-1)(1)} = \frac{2-1}{1-2} = \frac{1}{-1} = -1 = m$

(iii) $y-y_1 = m(x-x_1)$

$y-1 = -1(x+1)$

$y-1 = -x-1$

$\boxed{y=-x}$ or $\boxed{x+y=0}$