

## Series + Sequence

$$\textcircled{1} \quad \underline{80000}, \quad \underline{\quad}, \quad \underline{\quad}, \quad \underline{\quad}, \quad \underline{117128}$$

$$a = 80000$$

$$t_n = ar^{n-1}$$

$$t_5 = 117128$$

$$\frac{117128}{80000} = \frac{80000r^{5-1}}{80000}$$

$$n = 5$$

$$r = ?$$

$$\text{AROI} = ?$$

$$(1.4641)^{\frac{1}{4}} = (r^4)^{\frac{1}{4}}$$

$$\boxed{1.1 = r}$$

$$\text{AROI} = 100(1.1 - 1) = 10\%$$

## Series + Sequence

$$\textcircled{5} \quad a) \sum_{n=1}^5 n^2 + 1$$

$$= [(1)^2 + 1] + [(2)^2 + 1] + [(3)^2 + 1] + [(4)^2 + 1] + [(5)^2 + 1]$$

$$= 2 + 5 + 10 + 17 + 26$$

$$= 60$$

$$b) \sum_{n=1}^{\infty} 3 \left( \frac{1}{2} \right)^{n-1}$$

$$S_n = \frac{a}{1-r}$$

$$S_n = \frac{3}{1 - \frac{1}{2}}$$

$$a = 3$$

$$r = \frac{1}{2}$$

$$S_n = \frac{3}{\frac{2}{2} - \frac{1}{2}}$$

$$S_n = \frac{3}{\frac{1}{2}} = 3 \times 2 = \textcircled{6}$$

## Series + Sequence

$$\textcircled{5} \text{ c) } \underline{1} + 5 + 9 + \dots + \underline{77}$$

$\begin{array}{c} \vee \quad \vee \\ 4 \quad 4 \end{array}$

$$a = 1$$

$$t_n = 77$$

$$d = 4$$

① Find  $n$ .

$$t_n = a + (n-1)d$$

$$77 = 1 + (n-1)4$$

$$76 = 4(n-1)$$

$$19 = n-1$$

$$\boxed{20 = n}$$

② Find  $S_{20}$

$$S_{20} = \frac{20}{2} [1 + 77]$$

$$= 10(78)$$

$$\boxed{= 780}$$

$$\textcircled{5} \text{ b) } \sum_{n=1}^{\infty} \underline{3} \left( \frac{1}{2} \right)^{n-1} = 3 + \frac{3}{2} + \frac{3}{4} + \dots$$

$\begin{array}{c} \vee \quad \vee \\ \frac{1}{2} \quad \frac{1}{2} \end{array}$

$$a = 3$$

$$r = \frac{1}{2}$$

$$S_n = \frac{3}{1 - \frac{1}{2}} = \frac{3}{\frac{1}{2}} = 3 \cdot 2 = \textcircled{6}$$

$$\textcircled{4} \text{ c) } \lim_{n \rightarrow \infty} (-1)^{n+1} n^2$$

$$= 1, -4, 9, -16, 25, -36, 49, -64, 81, -100$$

= Diverging (has no limit)

## Series &amp; Sequence

⑥ Find "a", "r", and  $S_5$   $t_n = ar^{n-1}$   
 $\frac{1}{9} \div 9 = \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$

$$\begin{array}{l|l} \underline{t_3} = 9 & \underline{t_7} = \frac{1}{9} \\ t_3 = ar^{3-1} & t_7 = ar^{7-1} \\ \underline{t_3} = ar^2 & \underline{t_7} = ar^6 \\ 9 = ar^2 & \frac{1}{9} = ar^6 \\ ar^2 = 9 & ar^6 = \frac{1}{9} \end{array}$$

Elimination

$$\begin{array}{l} ar^6 = \frac{1}{9} \\ ar^2 = 9 \\ r^4 = \frac{1}{81} \end{array} \quad \left. \begin{array}{l} ar^2 = 9 \\ a\left(\frac{1}{3}\right)^2 = 9 \\ a\left(\frac{1}{9}\right) = 9 \end{array} \right\} \begin{array}{l} \cancel{r} \cdot \frac{a}{\cancel{9}} = 9 \cdot 9 \\ \boxed{a = 81} \end{array}$$

when  $r = \frac{1}{3}$ 

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{81\left(\left(\frac{1}{3}\right)^5 - 1\right)}{\frac{1}{3} - 1}$$

$$S_5 = 81 \left( \frac{\frac{1}{243} - \frac{243}{243}}{\frac{1}{3} - \frac{3}{3}} \right)$$

$$S_5 = 81 \left( \frac{-\frac{242}{243}}{-\frac{2}{3}} \right)$$

$$S_5 = 81 \left( \frac{121}{243} \right) \left( \frac{-3}{-2} \right)$$

$$S_5 = 121$$

when  $r = -\frac{1}{3}$ 

$$S_5 = \frac{81\left(\left(-\frac{1}{3}\right)^5 - 1\right)}{-\frac{1}{3} - 1}$$

$$S_5 = 81 \left( \frac{-\frac{1}{243} - \frac{243}{243}}{-\frac{1}{3} - \frac{3}{3}} \right)$$

$$S_5 = 81 \left( \frac{\frac{61}{243}}{-\frac{4}{3}} \right) \left( \frac{-3}{4} \right)$$

$$S_5 = 61$$



## Series &amp; Sequence

$$t_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\textcircled{1} \quad t_{12} = 15$$

$$t_{12} = a + (12-1)d$$

$$t_{12} = a + 11d$$

$$15 = a + 11d$$

$$a + 11d = 15$$

$$S_{15} = 105$$

$$S_{15} = \frac{15}{2}[2a + (15-1)d]$$

$$S_{15} = 7.5[2a + 14d]$$

$$S_{15} = 15a + 105d$$

$$105 = 15a + 105d$$

$$15a + 105d = 105$$

$$a + 7d = 7$$

## Elimination

$$a + 11d = 15$$

$$\Leftrightarrow a + 7d = 7$$

$$\frac{4d = 8}{4 \quad 4}$$

$$d = 2$$

↳

$$a + 7d = 7$$

$$a + 7(2) = 7$$

$$a + 14 = 7$$

$$a = -7$$

$$t_1 = -7$$

$$t_2 = -5$$

$$t_3 = -3$$

## Series &amp; Sequence

—,  $\frac{1}{5}$ , —, —,  $25$ , —, —

$$t_2 = \frac{1}{5}, \quad t_5 = 25$$

$$t_2 = ar^{2-1}, \quad t_5 = ar^{5-1}$$

$$t_2 = ar^1, \quad t_5 = ar^4$$

$$\frac{1}{5} = ar, \quad 25 = ar^4$$

$$ar = \frac{1}{5}, \quad ar^4 = 25$$

$$\frac{ar^4 = 25}{ar = \frac{1}{5}}$$

$$r^3 = 125$$

$$r = 5$$

$$ar = \frac{1}{5}$$

$$a(5) = \frac{1}{5}$$

$$5a = \frac{1}{5}$$

$$a = \frac{1}{25}$$

## Permutations + Combinations

### binomial theorem

- used to expand  $(x + y)^n$ ,  $n \in \mathbb{N}$
- each term has the form  ${}_n C_k (x)^{n-k} (y)^k$ , where  $k + 1$  is the term number

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x + y)^n = {}_n C_0 (x)^n (y)^0 + {}_n C_1 (x)^{n-1} (y)^1 + {}_n C_2 (x)^{n-2} (y)^2 + \dots \\ + {}_n C_{n-1} (x)^1 (y)^{n-1} + {}_n C_n (x)^0 (y)^n$$

In this chapter, all binomial expansions will be written in descending order of the exponent of the first term in the binomial.

The following are some important observations about the expansion of  $(x + y)^n$ , where  $x$  and  $y$  represent the terms of the binomial and  $n \in \mathbb{N}$ :

- the expansion contains  $n + 1$  terms
- the number of objects,  $k$ , selected in the combination  ${}_n C_k$  can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term,  $t_{k+1}$ , has the form

$${}_n C_k (x)^{n-k} (y)^k$$

↑  
the same

- the sum of the exponents in any term of the expansion is  $n$

## Permutations + Combinations

- ① 3 digit # from 0, 1, 2, 8, and 9  
with no repetitions

$$\underline{4} \times \underline{4} \times \underline{3} = 48 \quad C$$

↑  
can't  
use 0  
as digit  
one

$$\textcircled{15} \left(y - \frac{2}{y^2}\right)^5 \quad n=5$$

$$x=y$$

$$y = \frac{-2}{y^2}$$

$${}^5C_0(y)^5\left(\frac{-2}{y^2}\right)^0 + {}^5C_1(y)^4\left(\frac{-2}{y^2}\right)^1 + {}^5C_2(y)^3\left(\frac{-2}{y^2}\right)^2 + {}^5C_3(y)^2\left(\frac{-2}{y^2}\right)^3 + {}^5C_4(y)^1\left(\frac{-2}{y^2}\right)^4 + {}^5C_5(y)^0\left(\frac{-2}{y^2}\right)^5$$

$$1(y^5)(1) + 5(y^4)\left(\frac{-2}{y^2}\right) + 10(y^3)\left(\frac{4}{y^4}\right) + 10(y^2)\left(\frac{-8}{y^6}\right) + 5(y)\left(\frac{16}{y^8}\right) + 1(1)\left(\frac{-32}{y^{10}}\right)$$

$$y^5 - \frac{10y^4}{y^2} + \frac{40y^3}{y^4} - \frac{80y^2}{y^6} + \frac{80y}{y^8} - \frac{32}{y^{10}}$$

$$\boxed{y^5 - 10y^2 + \frac{40}{y} - \frac{80}{y^4} + \frac{80}{y^7} - \frac{32}{y^{10}}}$$

## Permutations + Combinations

$$\textcircled{5} \quad (\underline{2x^2} + \underline{3y})^7 \rightarrow 3^{\text{rd}} \text{ term}$$

$${}^7C_2 (2x^2)^5 (3y)^2$$

$$21 (32x^{10}) (9y^2)$$

$$\boxed{6048x^{10}y^2}$$

## Permutations + Combinations

$$\textcircled{a} \left(x^2 - \frac{x}{2}\right)^5$$

$${}_5C_0(x^2)^5\left(\frac{-x}{2}\right)^0 + {}_5C_1(x^2)^4\left(\frac{-x}{2}\right)^1 + {}_5C_2(x^2)^3\left(\frac{-x}{2}\right)^2 + {}_5C_3(x^2)^2\left(\frac{-x}{2}\right)^3 + {}_5C_4(x^2)^1\left(\frac{-x}{2}\right)^4 + {}_5C_5(x^2)^0\left(\frac{-x}{2}\right)^5$$

$$1(x^{10})(1) + 5(x^8)\left(\frac{-x}{2}\right) + 10(x^6)\left(\frac{x^2}{4}\right) + 10(x^4)\left(\frac{-x^3}{8}\right) + 5(x^2)\left(\frac{x^4}{16}\right) + 1(1)\left(\frac{-x^5}{32}\right)$$

$$x^{10} - \frac{5x^9}{2} + \frac{5x^8}{2} - \frac{5x^7}{4} + \frac{5x^6}{16} - \frac{x^5}{32}$$

## Permutations + Combinations

Expand:  $(x - \frac{1}{2})^4$ 

$${}^4C_0(x)^4(\frac{1}{2})^0 + {}^4C_1(x)^3(\frac{1}{2})^1 + {}^4C_2(x)^2(\frac{1}{2})^2 + {}^4C_3(x)^1(\frac{1}{2})^3 + {}^4C_4(x)^0(\frac{1}{2})^4$$

$$(1)(x^4)(1) + (4)(x^3)(\frac{1}{2}) + (6)(x^2)(\frac{1}{4}) + (4)(x)(\frac{1}{8}) + (1)(1)(\frac{1}{16})$$

$$x^4 - \frac{4x^3}{2} + \frac{6x^2}{4} - \frac{4x}{8} + \frac{1}{16}$$

$$x^4 - 2x^3 + \frac{3x^2}{2} - \frac{1}{2}x + \frac{1}{16}$$

## Permutations + Combinations

④ c) Case 1 (All Black)

$${}_{26}C_5 = \boxed{65\,780}$$

Case 2 (4 Black + 1 Red)

$${}_{26}C_4 \times {}_{26}C_1 = 14\,950 \times 26 = \boxed{388\,700}$$

Case 3 (3 black + 2 red)

$${}_{26}C_3 \times {}_{26}C_2 = 2600 \times 325 = \boxed{845\,000}$$

$$\text{Total} = 845\,000 + 388\,700 + 65\,780 = \boxed{1\,299\,480}$$



## Permutations + Combinations

⑤ 14 Letters

- 5 are burnt
- 9 are good

$$3 \text{ good} + 2 \text{ bad}$$
$${}^9C_3 \times {}^5C_2$$

$$84 \times 10$$

$$840$$

Limits

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

## Limits

$$\textcircled{1} \text{ c) } \lim_{a \rightarrow b} \frac{\underline{(a+2b)^2} - \underline{9b^2}}{a-b}$$

$$\lim_{a \rightarrow b} \frac{(a+2b+3b)(a+2b-3b)}{(a-b)}$$

$$\lim_{a \rightarrow \underline{b}} \frac{\underline{(a+5b)} \cancel{(a-b)}}{\cancel{(a-b)}} = 6b$$

$$\textcircled{1} \text{ f) } \lim_{x \rightarrow 0} \frac{\cancel{5(x+5)} \frac{1}{x+5} - \frac{1}{5} \cancel{5(x+5)}}{(x^2+5x)(5(x+5))} \quad \text{CD: } 5(x+5)$$

$$\lim_{x \rightarrow 0} \frac{5 - 1(x+5)}{x(x+5)(5(x+5))}$$

$$\lim_{x \rightarrow 0} \frac{5-x-5}{5x(x+5)^2}$$

$$\lim_{x \rightarrow \underline{0}} \frac{\cancel{-x}}{5x \cancel{(x+5)}^2} = \frac{-1}{5(25)} = -\frac{1}{125}$$

## Limits

$$\textcircled{2} \text{ Let } f(x) = \begin{cases} -x^2 + 2, & \text{if } x < 1 \\ 1, & \text{if } x = 1 \\ -2x + 3, & \text{if } 1 < x \leq 3 \\ -3, & \text{if } x > 3 \end{cases}$$

$$-x^2 + 2$$

x	y
1	1
0	2
-1	1
-2	-2

$$1$$

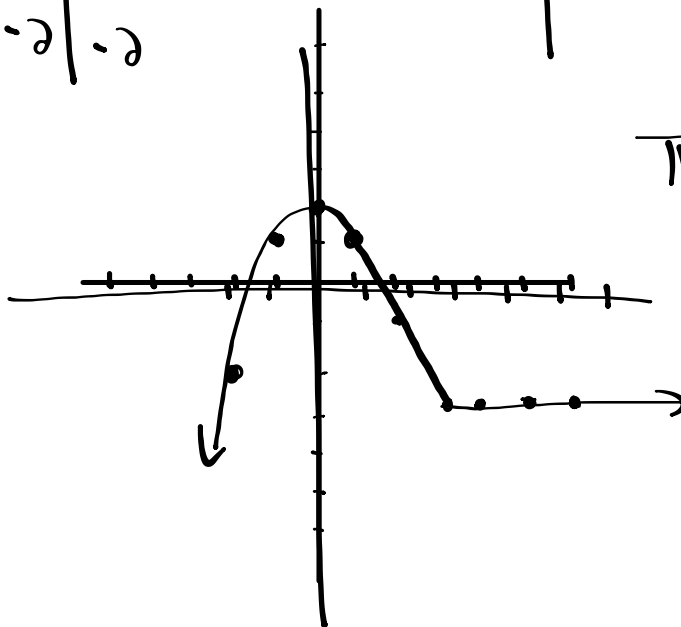
x	y
1	1

$$-2x + 3$$

x	y
1	1
2	-1
3	-3

$$-3$$

x	y
3	-3
4	-3
5	-3
6	-3



The function is continuous

## Limits

$$\lim_{x \rightarrow 0} \frac{\cancel{3(x+3)} \frac{1}{\cancel{x+3}} - \frac{1}{\cancel{3}} \cancel{3(x+3)}}{x \cdot 3(x+3)}$$

$$\text{CD: } 3(x+3)$$

$$\lim_{x \rightarrow 0} \frac{3 - (x+3)}{3x(x+3)}$$

$$\lim_{x \rightarrow 0} \frac{3 - x - 3}{3x(x+3)}$$

$$\lim_{x \rightarrow 0} \frac{\cancel{-x}}{\underline{3x}(\underline{x+3})} = \frac{-1}{3(3)} = \frac{-1}{9}$$

## Differentiation

Product:  $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$

$$y = \underbrace{(3x^2-5)}_{f(x)} \underbrace{(4x^3+3x)}_{g(x)}$$

$$y' = 6x(4x^3+3x) + (3x^2-5)(12x^2+3)$$

$$y' = 24x^4 + 18x^2 + 36x^4 + 9x^2 - 60x^2 - 15$$

$$y' = 60x^4 - 33x^2 - 15$$

Quotient:  $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$$y = \frac{x+2}{3x+5}$$

$$y' = \frac{1(3x+5) - 3(x+2)}{(3x+5)^2}$$

$$y' = \frac{6x^2 + 10x - 3x^2 - 6}{(3x+5)^2} = \frac{3x^2 + 10x - 6}{(3x+5)^2}$$

Chain:  $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$$y = \sqrt{4x^3-6x} = (4x^3-6x)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(4x^3-6x)^{-\frac{1}{2}}(12x^2-6)$$

$$y' = \frac{12x^2-6}{2(4x^3-6x)^{\frac{1}{2}}} = \frac{3(2x^2-1)}{2\sqrt{4x^3-6x}}$$

$$\frac{6x^2-3}{\sqrt{4x^3-6x}}$$

Combo:

$$y = (3x^2+5)^3 \sqrt{4x-2} = \underbrace{(3x^2+5)^3}_{f(x)} \underbrace{(4x-2)^{\frac{1}{2}}}_{g(x)}$$

$$y' = 3(3x^2+5)^2(6x)(4x-2)^{\frac{1}{2}} + (3x^2+5)^3 \left(\frac{1}{2}\right)(4x-2)^{-\frac{1}{2}}(4)$$

$$y' = 18x(3x^2+5)^2(4x-2)^{\frac{1}{2}} + 2(3x^2+5)^3(4x-2)^{-\frac{1}{2}}$$

$$y' = 2(3x^2+5)^2(4x-2)^{-\frac{1}{2}} [9x(4x-2) + (3x^2+5)]$$

$$y' = 2(3x^2+5)^2(4x-2)^{-\frac{1}{2}}(39x^2-18x+5)$$

$$y' = \frac{2(3x^2+5)^2(39x^2-18x+5)}{\sqrt{4x-2}}$$

$$\frac{(3x^2+5)^2}{(3x^2+5)^2} = (3x^2+5)^0 = 1 \quad \left| \quad \frac{(4x-2)^{\frac{1}{2}}}{(4x-2)^{\frac{1}{2}}} = (4x-2)^{\frac{1}{2}-\frac{1}{2}} = (4x-2)^0 = 1\right.$$

## Differentiation

$$f(x) = 3x^2 + 2x - 7$$

$$f(x+h) = 3(x+h)^2 + 2(x+h) - 7$$

$$= 3(x^2 + 2xh + h^2) + 2x + 2h - 7$$

$$= 3x^2 + 6xh + 3h^2 + 2x + 2h - 7$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 2x + 2h - 7 - (3x^2 + 2x - 7)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 2h}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(6x + 3h + 2)}{h} = 6x + 2$$

## Differentiation

$$f(x) = \frac{(3x^2+5)^3}{\sqrt{2x-7}}$$

$$\frac{f'g - fg'}{g^2}$$

$$f'(x) = \frac{\overbrace{3(3x^2+5)^2}^{f'} \overbrace{(6x)}^g \overbrace{(2x-7)^{-1/2}}^{g'}}{\underbrace{[\sqrt{2x-7}]^2}} - \overbrace{(3x^2+5)^3}^f \overbrace{\left(\frac{1}{2}\right)}^{f'} \overbrace{(2x-7)^{-1/2}}^{g'}}$$

$$f'(x) = \frac{18x(3x^2+5)^2(2x-7)^{1/2} - (3x^2+5)^3(2x-7)^{-1/2}}{\dots} \quad \leftarrow \text{Factor}$$

$$f'(x) = \frac{(3x^2+5)^2(2x-7)^{-1/2} \left[ \overset{2x-7}{18x(2x-7)} - (3x^2+5) \right]}{(2x-7)}$$

$$f'(x) = \frac{(3x^2+5)^2(33x^2-126x-5)}{(2x-7)^{3/2}}$$



## Differentiation

$$\textcircled{3} \quad y = \sqrt[7]{2x^2 + \sqrt{x^2 - 8x} \sqrt{3-x}} = \left[ 2x^2 + (x^2 - 8x(3-x))^{\frac{1}{2}} \right]^{\frac{1}{7}}$$

$$y' = \frac{1}{7} \left[ 2x^2 + (x^2 - 8x(3-x))^{\frac{1}{2}} \right]^{-\frac{6}{7}} \left[ 4x + \frac{1}{2} (x^2 - 8x(3-x))^{-\frac{1}{2}} (2x^2 - (8(3-x)^{\frac{1}{2}} + 8x(\frac{1}{2})(3-x)^{-\frac{1}{2}}(1))) \right]$$

## Derivatives Exam Review!

$$\textcircled{4} \text{ b) } y = \frac{\sqrt{x}}{3+x^2} = \frac{x^{1/2}}{(3+x^2)}$$

$$y' = \frac{(3+x^2)\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2}(2x)}{(3+x^2)^2}$$

$$y' = \frac{\frac{3}{2}x^{-1/2} + \frac{1}{2}x^{3/2} - 2x^{3/2}}{2(3+x^2)^2}$$

$$y' = \frac{3x^{-1/2} + x^{3/2} - 4x^{3/2}}{2(3+x^2)^2}$$

$$y' = \frac{x^{-1/2}(3+x^2-4x^2)}{2(3+x^2)^2}$$

$$y' = \frac{3-3x^2}{2x^{1/2}(3+x^2)^2} = \frac{3(1-x^2)}{2\sqrt{x}(3+x^2)^2}$$

